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Thermal Relaxation of a Two Dimensional

Plasma in a dc Magnetic Field

Part I: Theory

by

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ABSTRACT

A theory is presented for the rate of thermal relaxation of a two dimensional plasma in a strong uniform dc magnetic field. The Vahala - Montgomery kinetic description is completed by providing a cut off time for the time of interaction of two particles which contribute to the collision term. The kinetic equation predicts that thermal relaxation occurs as a function of the dimensionless time $(\omega_p t) (\omega_p/\Omega) (n_o\lambda_D^2)^{-1/2}$, where ω_p is the plasma frequency, Ω is the gyrofrequency, and $n_o\lambda_D^2$ is the number of particles per Debye square. By contrast, in the absence of an external magnetic field, a two dimensional plasma relaxes as a function of $(\omega_p t) (n_o\lambda_D^2)^{-1}$.

1. Introduction

Two dimensional electrostatic models of a plasma in a strong magnetic field have received considerable attention over the last three years (see, e.g., Taylor and McNamara 1971; Vahala and Montgomery 1971; Vahala 1972; Montgomery and Tappert 1972; Montgomery 1972 a,b; Joyce and Montgomery 1972; Taylor 1972; Joyce and Montgomery 1973). Much of the interest has revolved around the guiding centre model, in which particle positions are advanced by their self-consistent electrostatic $\underset{\sim}{E} \times \underset{\sim}{B}$ drifts. The present article, however, deals with the finite gyroradius case, in which the guiding centre approximation is not made.

Of primary concern is the rate of thermal relaxation of the model, a phenomenon which does not occur at all in the guiding centre limit (Vahala and Montgomery, 1971). In particular, we are interested in the dependence of the relaxation rate on the various dimensionless parameters of the problem such as the number of particles per Debye square and the ratio of gyrofrequency to plasma frequency. In this paper (Part I) various theoretical considerations relevant to the thermal relaxation process are developed. In the companion paper (Part II), these predictions are tested by numerical simulation.

2. The Basic Model

As usual, the particles are taken to be very long rods of length ℓ, charge $-e$, mass m, which remain aligned paralled to a uniform dc magnetic field $\underset{\sim}{B} = B\hat{b}$ that points in the z-direction.

The position $x_i(t)$ of the **ith** rod is a two-component vector in the xy plane. The potential energy of interaction of two rods is taken to be $- (2e^2/\ell) \ell n \mid x_i(t) - x_j(t) \mid$ at time t. A uniform immobile positive background charge provides overall charge neutrality. The equations of motion of the ith charge are $dx_i(t)/dt = v_i(t)$, $dv_i(t)/dt = (- e/m)(E + v_i \times B/c)$. $E = - \partial\phi/\partial x$ is the electric field and is determined through Poisson's equation and whatever boundary conditions may obtain.

Equivalently, the dynamics may be specified by the <u>Klimontovich equation</u> (see e.g., Dupree 1963, 1964; Montgomery, 1971):

$$\left\{ \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial x} - \frac{e}{m} \left(E + \frac{v}{c} \times B\right) \cdot \frac{\partial}{\partial v} \right\} f \equiv \frac{Df}{Dt} = 0 \quad , \qquad (1)$$

where the exact distribution f is

$$f = n_o^{-1} \sum_i \delta(x - x_i(t))\delta(v - v_i(t)) \quad , \qquad (2)$$

and n_o is the average number of rods per unit area. E is given by the solution of

$$\frac{\partial}{\partial x} \cdot E = - \frac{\partial^2\phi}{\partial x^2} = \frac{4\pi e n_o}{\ell} \left[1 - \int f dv\right] \quad . \qquad (3)$$

Even though the distribution function f has been introduced, there is as yet nothing of statistical mechanics in the formulation.

3. Statistical Formulation

Statistical mechanics may be introduced by considering the initial values of the x_i, v_i $(x_i(0) \equiv x_{io}$, $v_i(0) \equiv v_{io}$, say) to be statistically distributed in their phase spaces according to some probability distribution law. (For equilibrium situations, this will be the canonical ensemble of Gibbs, but otherwise it will not be.) We indicate expectation values or ensemble averages by a bracket $\langle \ \rangle$, and write $f = \langle f \rangle + \delta f$, $E = \langle E \rangle + \delta E$. For spatially uniform ensembles, we will have $\langle E \rangle = 0$ and $\partial \langle f \rangle / \partial x = 0$.

For spatially uniform and gyrotropic $\langle f \rangle$'s, $v \times B \cdot \partial \langle f \rangle / \partial v = 0$, and ensemble averaging Eq. (1) will give

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{e}{m} \frac{\partial}{\partial v} \cdot \langle Ef \rangle \quad , \tag{4}$$

where

$$E = - \frac{2n_o e}{\ell} \int dx' dv' \frac{x - x'}{|x - x'|^2} f(x', v', t) \quad , \tag{5}$$

and the point $x = x'$ is to be omitted from the integral in Eq. (5).

It is clear that what is required for the evaluation of the right hand side of Eq. (4) is

$$\langle ff' \rangle \equiv \langle f(\underline{x}, \underline{v}, t)f(\underline{x}', \underline{v}', t) \rangle$$

$$= \sum_{\substack{i,j \\ i \neq j}} \langle \delta(\underline{x} - \underline{x}_i(t))\delta(\underline{v} - \underline{v}_i(t))\delta(\underline{x}'- \underline{x}_j(t))\delta(\underline{v}'- \underline{v}_j(t)) \rangle$$

$$(6)$$

Eqs. (1) through (6) are exact, and it is well known that approximations and hypotheses must be made before Eq. (4) will lead to a significant or tractable kinetic description. By this we mean that the right hand side of Eq. (4) is expressed as a functional of $\langle f \rangle$ alone for which we may prove conservation laws, an H-theorem, and so forth. The program through which these hypotheses and approximations are developed is too lengthy to expose here, and reference may be made to textbooks for the relevant background (e.g., Montgomery 1967, 1971). All the approximations and assumptions we shall make are standard except one, and this one will be given special attention when it occurs in Section 4.

We shall be interested in the Fokker-Planck limit, or the limit in which the potential energy of interaction between two typical particles is taken to be much less than the average kinetic energy per particle, θ. This is, in effect, a perturbation expansion in the dimensionless ratio $e^2/\ell\theta$. It is justified by the smallness

of $(n_o \lambda_D^2)^{-1}$ (Montgomery, 1972 b). The fact that the right hand side

of Eq. (4) is of second order in this parameter implies that the

evolution of $\langle f \rangle$ is very slow compared to the explicit time develop-

ment of $\langle ff' \rangle$. Following what is essentially the Bogolyubov recipe

(Bogolyubov 1946; Dupree 1961; Montgomery 1971), the asymptotic

$t \to \infty$ form of $\langle Ef \rangle$ may be taken while holding $\langle f \rangle$ fixed. The weak

coupling expansion implies that we insert the unperturbed (i.e., non-

interacting) orbits into Eq. (1) and iterate, to find successive ap-

proximations to f. The Fokker-Planck equation results if we stop

the iteration at the stage where the right hand side of Eq. (4) first

fails to vanish. Ensemble averages $\langle \ \rangle$ are carried out by ignoring

initial correlations:

$$\langle A \rangle = \int \prod_i \left\{ \frac{dx_{io} dv_{io}}{V} \langle f(v_{io}) \rangle \right\} \quad A \quad . \tag{7}$$

Here, A stands for any phase function (including ff' or fE), $V = L^2$

is the configuration-space volume, and the product $\prod_i$ runs over all

the charges. Higher order corrections to $\langle ff' \rangle$ or $\langle fE \rangle$ are computed

in the same way, with corrections to f and E being calculated <u>before</u>

the averages $\langle \ \rangle$ are performed.

4. The Kinetic Equation

The unperturbed orbits $\underset{\sim}{x}_i(t)$, $\underset{\sim}{v}_i(t)$ are

$$\underset{\sim}{x}_i(t) = \underset{\sim}{x}_{io} + \underset{\sim}{v}_{io} \frac{\sin \Omega t}{\Omega}$$

$$- \hat{b} \times \underset{\sim}{v}_{io} \frac{\cos \Omega t - 1}{\Omega}$$

$$\underset{\sim}{v}_i(t) = \underset{\sim}{v}_{io} \cos \Omega t + \hat{b} \times \underset{\sim}{v}_{io} \sin \Omega t \quad , \tag{8}$$

where $\Omega \equiv |eB_o/mc|$. Use of Eq. (8) and Eq. (7) in Eqs. (6) and (4) leads, after some lengthy algebra, to the following kinetic equation for $\langle f \rangle$ (Vahala and Montgomery 1971; Vahala 1972):

$$\frac{\partial \langle f \rangle}{\partial t} =$$

$$- \frac{\partial}{\partial \underset{\sim}{v}} \cdot \int d\underset{\sim}{v}' \underset{\approx}{Q}(\underset{\sim}{v}, \underset{\sim}{v}') \cdot \left(\frac{\partial}{\partial \underset{\sim}{v}} - \frac{\partial}{\partial \underset{\sim}{v}'} \right) \langle f(\underset{\sim}{v}) \rangle \langle f(\underset{\sim}{v}') \rangle \tag{9}$$

where the dyadic $\underset{\approx}{Q}$ is given by

$$\underset{\approx}{Q} = - T_o \frac{4\pi m_o e^4}{m^2 \ell^2} \frac{(\underset{\sim}{v} - \underset{\sim}{v}') \times \hat{b} \ (\underset{\sim}{v} - \underset{\sim}{v}') \times \hat{b}}{|\underset{\sim}{v} - \underset{\sim}{v}'|^2} \quad . \tag{10}$$

Eq. (9) has been written in standard Balescu-Lenard form.
The time T_0 is a cut off which has remained up to this point an
undetermined, troublesome constant. (An H-theorem and conservation
laws follow from Eq. (9) no matter what T_0 is, but relaxation times
scale directly as T_0^{-1}.) T_0 has the physical significance of being
the interaction time of two particles along their unperturbed
trajectories. Because of the absence, in two dimensions, of motion
parallel to the magnetic field, the unperturbed motions are strictly
periodic and T_0 is infinite. Until now, a wholly satisfactory
argument for assigning T_0 a finite value has been lacking (Vahala and
Montgomery 1971; Montgomery 1972 b). We are now able to determine
T_0.

Detailed inspection of the multiple integrals which contribute
to $\underset{\approx}{Q}$ reveals that for T_0 large, the only regions of phase space
which lead to contributions to $\underset{\approx}{Q}$ are those which correspond to over-
lapping gyroradii. It is only because we have used a perturbation
scheme based upon pairwise interactions that overlapping gyroradii
continue to overlap for all time. For the true orbits, gyrocenters
will slowly diffuse apart due to long wavelength electric field
fluctuations of the Taylor-McNamara (1971) type. The effect of these
can be introduced in a non-rigorous (but to us wholly convincing) way
by determining T_0 to be that time required to diffuse two gyrocenters
a thermal gyroradius apart:

$$D_{T.M.} = \frac{\theta}{m}\left(\frac{mc}{eB}\right)^2 \frac{1}{T_o} \tag{11}$$

where $D_{T.M.}$ is the Taylor-McNamara diffusion coefficient for one species:

$$D_{T.M.} = \frac{c\theta}{eB} \frac{1}{\sqrt{n_o \lambda_D^2}} \sqrt{\ell n(L/2\pi\lambda D)} \quad . \tag{12}$$

$\lambda_D^2 \equiv \ell\theta/4\pi n_o e^2$ is the square of the Debye length, and θ is the mean kinetic energy per particle. Eqs. (11) and (12) remove the ambiguities that existed in Eq. (9), for now

$$T_o = \sqrt{n_o \lambda_D^2} \Big/ \left(\Omega \sqrt{\ell n(L/2\pi\lambda D)}\right) \tag{13}$$

Except for a weak (logarithmic) dependence on the plasma volume, the thermal relaxation is predicted to occur as a function of the dimensionless time

$$(\omega_p t)(\omega_p/\Omega)\Big/\sqrt{n_o \lambda_D^2} \quad ,$$

where $\omega_p \equiv 4\pi n_o e^2/\ell m$ defines the plasma frequency. It is well known

and has been demonstrated numerically (Montgomery and Nielson, 1970) that the thermal relaxation of a two dimensional unmagnetized plasma occurs as a function of the dimensionless time

$$(\omega_p t)/n_o \lambda_D^2 \quad .$$

(A theoretical discussion is due to Yoo and Abraham-Shrauner 1973.) There are thus two clear-cut predictions for the rate at which thermal relaxation occurs for a strongly-magnetized plasma which differ from anything suggested previously: (1) the relaxation time is proportional to the magnetic field strength; and (2) the relaxation time is proportional to $\sqrt{n_o \lambda_D^2}$, the reciprocal of the square root of the plasma parameter. Both these predictions are tested numerically in Part II.

A condition for the validity of the use of the cut off (13) in Eqs. (9) and (10) is that two thermal particles shall separate in a time T_o which is short compared to a relaxation time for $\langle f \rangle$. It has been seen in the last paragraph that the latter time is of the order of $\omega_p^{-1} \left(\Omega/\omega_p \right) \sqrt{n_o \lambda_D^2}$. The relevant inequality is therefore

$$\left(\frac{\Omega}{\omega_p} \right)^2 \gg \frac{1}{\sqrt{\ell n(L/2\pi\lambda D)}} \tag{14}$$

which is satisfied for strong enough $\underset{\sim}{B}$.

5. Definition of Relaxation Times

Eqs. (9) and (13) imply a monotonic decrease of Boltzmann's

$H(t) = \int d\underset{\sim}{v} \langle f \rangle \ell n \langle f \rangle$ to a minimum value consistent with conservation of

the initial values of $\int d\underset{\sim}{v} \langle f \rangle$, $\int d\underset{\sim}{v} \underset{\sim}{v} \langle f \rangle$, and $\int d\underset{\sim}{v} v^2 \langle f \rangle$. For an

initial value of $\langle f \rangle$, $H(\infty)$ is uniquely determined and a relaxation

time can be defined as the time required for $H(t)$ to reach some

value between $H(o)$ and $H(\infty)$. It is by numerically simulating the

particle dynamics and periodically computing H that relaxation times

are measured in Part II and compared with these predictions. The

$\underset{\sim}{B} = 0$ results were reported some time ago by Montgomery and Nielson

(1970).

It will be seen in Part II that the predictions that the

relaxation times scale as Ω/ω_p and $(n_o \lambda_D^2)^{\pm 1/2}$ are quite well fulfilled

above certain values of Ω/ω_p.

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